Exercise 17

A circular membrane in space lies over the region $x^2 + y^2 \leq a^2$. The maximum z component of points in the membrane is b. Assume that (x, y, z) is a point on the membrane. Show that the corresponding point (r, θ, z) in cylindrical coordinates satisfies the conditions $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$, $|z| \leq b$.

Solution

Switch to cylindrical coordinates (r, θ, z) using the standard transformation.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

As a result, the region where the membrane is over becomes

$$x^2 + y^2 \leq a^2$$

$$(r\cos\theta)^2 + (r\sin\theta)^2 \leq a^2$$

$$r^2\cos^2\theta + r^2\sin^2\theta \leq a^2$$

$$r^2(\cos^2\theta + \sin^2\theta) \leq a^2$$

$$r^2 \leq a^2$$

$$r \leq a$$

Therefore, since r is nonnegative, $0 \leq r \leq a$. Since the formulas for x and y are in terms of $\cos\theta$ and $\sin\theta$, respectively, and these are 2π-periodic functions, $0 \leq \theta \leq 2\pi$. z is the same in cylindrical coordinates as it is in Cartesian coordinates, so if the membrane never goes higher than 8 or lower than -8,

$$-8 \leq z \leq 8,$$

that is

$$|z| \leq 8.$$